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Correlation of coordinate transformation parameters

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Abstract: Coordinate transformation parameters between two spatial Cartesian coordinate systems can be solved from the positions of non-colinear corresponding points. Based on the characteristics of translation, rotation and zoom components of the transformation, the complete solution is divided into three steps. Firstly, positional vectors are regulated with respect to the centroid of sets of points in order to separate the translation components. Secondly, the scale coefficient and rotation matrix are derived from the regulated positions independently and correlations among transformation model parameters are analyzed. It is indicated that this method is applicable to other sets of non-position data to separate the respective attributions for transformation parameters.

Key words: coordinate transformation model; Bursa model; orthnormal matrix; singular value decomposition (SVD); correlation

1 Introduction

Coordinate transformations include translation, rotation and zoom. On the basis of non-colinear corresponding point from two sets of spatial Cartesian coordinate system, inverse solution of transformation parameters is widely used in geodesy, photogrammetry, computer vision, etc.

Let $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ denote a pair of points, and they satisfy

$$\boldsymbol{X}_2 = \boldsymbol{T} + \lambda \boldsymbol{R}\boldsymbol{X}_1 \tag{1}$$

Where $\boldsymbol{T} = (\boldsymbol{T}_x, \boldsymbol{T}_y, \boldsymbol{T}_z)^{\mathrm{T}}$ and $\boldsymbol{R} = \boldsymbol{R}_x(\boldsymbol{R}_1)\boldsymbol{R}_y(\boldsymbol{R}_2)\boldsymbol{R}_z(\boldsymbol{R}_3)$, they are the rotation matrix and the translation vector, respectively; and λ is a scale.

The rotation parameter is expressed by Euler angles $(R_1, R_2, R_3)^{\mathrm{T}}$, which is called seven-parameter transformation model. For the small values of both the scale coefficient and the angles, which are common in space geodesy, we define

$$\lambda(1+\boldsymbol{D}) \text{ and } \boldsymbol{R} = (\boldsymbol{I} + \boldsymbol{R}^*) \tag{2}$$

and equation is expressed as (Bursa model)[1]:

$$\boldsymbol{X}_2 = \boldsymbol{T} + \boldsymbol{X}_1 + \boldsymbol{D}\boldsymbol{X}_1 + \boldsymbol{R}^*\boldsymbol{X}_1 \tag{3}$$

where

$$\boldsymbol{R}^* = \begin{pmatrix} 0 & -\boldsymbol{R}_3 & \boldsymbol{R}_2 \\ \boldsymbol{R}_3 & 0 & -\boldsymbol{R}_1 \\ -\boldsymbol{R}_2 & \boldsymbol{R}_1 & 0 \end{pmatrix}$$

is an anti-symmetric matrix, and $\boldsymbol{D}$ is scale factor.

When there are no less than three non-colinear corresponding points, the seven transformation parameters can be computed by the least-squares adjustment[2]. However, in practice, the strong correlations among transformation parameters often make some solved pa-

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rameters not accurate[3]. In reference[4], the parameters are divided into several groups, and the generalized correlation coefficients between the groups are analyzed. With a sampling area of 100 × 100 km^2, it is concluded that translation parameters have strong correlation with rotation and scale parameters, while the latter two are un-correlated. For generalized correlation coefficients are merely suitable to numerical validation, more rigorous certification are needed.

In this paper, the author proposes a new method to parameter inversion in coordinates transformation. Using this method, the consistent conclusions with reference[4] are drawn. Furthermore, this method can be used in some other kinds of non-positional analysis, such as baseline vectors or attitude matrix, in terms of fusion of multi-source information.

2 Position sensitivity to coordinate transformation parameters

There are two sets of position vectors $\{\boldsymbol{X}_{1i}\}$ *and* $\{\boldsymbol{X}_{2i}\}$ $(i=1,2,\cdots,N)$, and the parameters of the translation, rotation and zoom. In order to decouple the position sensitivity to coordinate transformation parameters, two corresponding centroids are introduced to derive the translation information, while the rotation and scale information are left in the normalized position sets.

2.1 Centroidal regulation

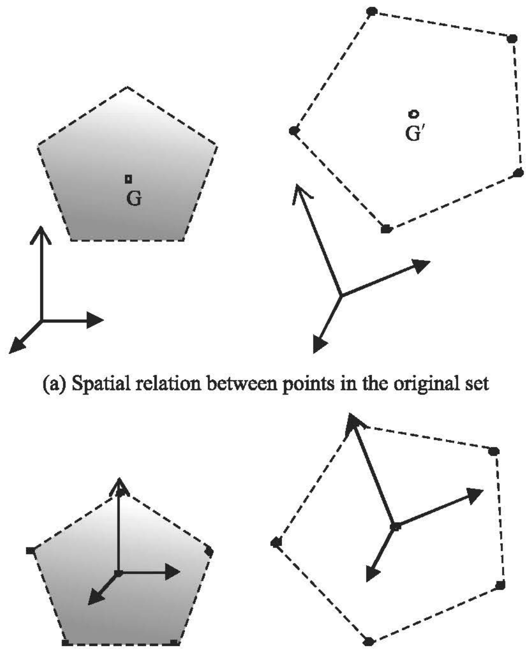
Suppose the position errors are independent, and have the equal precision, then define

$$\overline{\boldsymbol{X}}_{1G}=\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{X}_{1i},\ \overline{X}_{2G}=\frac{1}{N}\sum_{i=1}^{N}\boldsymbol{X}_{2i} \tag{4}$$

where, $\overline{\boldsymbol{X}}_{1G}$ and $\overline{\boldsymbol{X}}_{2G}$ are the centroid coordinates of $\{\boldsymbol{X}_{1i}\}$ and $\{\boldsymbol{X}_{2i}\}$, respectively.

$\{\overline{\boldsymbol{X}}_{1i}\}$ is supposed to completely match $\{\boldsymbol{X}_{2i}\}$ after the transformation if the coordinate errors are negligible, hence, the two centroids overlap. (Fig. 1(a)) and satisfy equation(1). It can be written as

$$\overline{\boldsymbol{X}}_{2G}=\boldsymbol{T}+\lambda\boldsymbol{R}\overline{\boldsymbol{X}}_{1G} \tag{5}$$



(a) Spatial relation between points in the original set

(b) Spatial relation between points in the original set with respect to centroid

Figure 1 Spatial relation between point set

Define: $\boldsymbol{X}'_{1i}=\boldsymbol{X}_{1i}-\overline{\boldsymbol{X}}_{1G},\ \boldsymbol{X}'_{2i}=\boldsymbol{X}_{2i}-\overline{X}_{2G}$ (6)

as the new sets of position with normalized coordinates. Substitute equations (4) – (6) into equation (1), the new relation between $\{\boldsymbol{X}'_{1i}\}$ and $\{\boldsymbol{X}'_{2i}\}$ is

$$\boldsymbol{X}'_{2i}=\lambda\boldsymbol{R}\boldsymbol{X}'_{1i} \tag{7}$$

The translation transformation is eliminated, as shown in figure 1(b).

Note that the position errors of the new set of points have changed slightly due to the errors in the original centroid coordinates. To equation (6), the error propagations satisfy

$$m_{X'_{1i}}=\sqrt{\frac{N-1}{N}}m_{X_{2i}},\ m_{X'_{2i}}=\sqrt{\frac{N-1}{N}}m_{X_{2i}} \tag{8}$$

Where m is the standard error. Obviously, the normalized coordinates will retain the same precisions with the original ones if the corresponding points are enough.

2.2 Dependence of translation on rotation and zoom

After the normalization with respect to centroid, the original sets of points have been divided into two parts, the centroids and the normalized coordinates, while the translation information is completely absorbed by the corresponding centroids (compared with figure 1(a) and (b)).

With the known estimates of both $\hat{\boldsymbol{R}}$ and $\hat{\lambda}$, we have the following relation for the translation parameters $\hat{\boldsymbol{T}}$ from equation (5)

$$\hat{\boldsymbol{T}} = \overline{\boldsymbol{X}}_{2G} - \hat{\lambda}\hat{\boldsymbol{R}}\overline{\boldsymbol{X}}_{1G} \tag{9}$$

Obviously, $\hat{\boldsymbol{T}}$ have a linear relation with the combination of $\hat{\boldsymbol{R}}$ and $\hat{\lambda}$. Therefore, if all the seven parameters are fitted by the least-squares, the strong correlations will exist in the estimated parameters of translation and the combination of rotation and zoom.

3 Decoupling of rotation and zoom

3.1 Scale estimation

Due to the module-retained property of rotation matrix, we have $|\boldsymbol{R}\boldsymbol{X}'_{1i}| = |\boldsymbol{X}'_{1i}|$. Now compute the 2-norm of equation (7), we have

$$\hat{\lambda} = \sqrt{\frac{\sum_{i=1}^{N} |\boldsymbol{X}'_{2i}|^2}{\sum_{i=1}^{N} |\boldsymbol{X}'_{1i}|^2}} \tag{10}$$

Which shows that the optimal scale is independent to the optimal rotation parameters.

3.2 Rotation matrix estimation with orthogonality constraint

There are several parameterizations of the rotation, but different parameters will affect the algorithm and accuracy[5]. Euler angles are usually adopted due to the clear geometrical meanings, while they need complex trigonometric arithmetic and may have potential mathematical singularity. It is convenient to construct linear equations by the elements of rotation matrix, but inevitably some nonlinear constraints are needed. Quaternion is the most applicable to the rigid dynamics if involved. Otherwise, the advantages will discount because the non-typical arithmetic of quaternion is considerably hard to understand. On the contrary, the rotation matrix itself to be taken into calculation as a whole is much more universal and acceptable.

Herein, the optimal rotation matrix $\hat{\boldsymbol{R}}$ is derived directly through singular value decomposition (SVD) as well as the properties of orthogonal and rotation matrices. Some other derivation methods based on the classical least-squares algorithm can be seen in references [2] and [6].

First, we construct the $3\times N$ position matrices from the normalized coordinates, namely, $\boldsymbol{X}$ and $\boldsymbol{Y}$

$$\begin{aligned} \boldsymbol{X} &= (\boldsymbol{X}'_{11}\cdots\boldsymbol{X}'_{1i}\cdots\boldsymbol{X}'_{1N}) \\ \boldsymbol{Y} &= (\boldsymbol{X}'_{21}\cdots\boldsymbol{X}'_{2i}\cdots\boldsymbol{X}'_{2N}) \end{aligned} \tag{11}$$

and they satisfy equation (7)

$$\boldsymbol{Y} = \lambda\boldsymbol{R}\boldsymbol{X} \tag{12}$$

Then construct the 3×3 matrix of $\boldsymbol{S}$ by

$$\boldsymbol{S} = \boldsymbol{X}\boldsymbol{Y}^{\mathrm{T}} \tag{13}$$

Let

$$\boldsymbol{X} = \boldsymbol{U}\Sigma\boldsymbol{V}^{\mathrm{T}} \tag{14}$$

Be an SVD of the real square matrix $\boldsymbol{S}$, where $\boldsymbol{U}$ and $\boldsymbol{V}$ are orthogonal matrices and Σ is diagonal.

From equation (14), we have

$$\begin{cases} \boldsymbol{U}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{V} = \Sigma \\ \boldsymbol{V}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{U} = \Sigma^{\mathrm{T}} \end{cases} \tag{15}$$

where $\Sigma = \Sigma^{\mathrm{T}}$, for they are the diagonal matrix, so

$$\boldsymbol{U}^{\mathrm{T}}\boldsymbol{S}\boldsymbol{V} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{U} \tag{16}$$

Now we can substitute the relation of $\boldsymbol{X}$ and $\boldsymbol{Y}$, the definition of matrix $\boldsymbol{S}$, and the equations (12) and (13), into equation (16)

$$\lambda\boldsymbol{U}^{\mathrm{T}}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{V} = \lambda\boldsymbol{V}^{\mathrm{T}}\boldsymbol{R}\boldsymbol{X}\boldsymbol{X}^{\mathrm{T}}\boldsymbol{U} \tag{17}$$

Note that the scale can be canceled in equation (17), and then left multiple by $\boldsymbol{U}$ and right multiple by $V^{\mathrm{T}}R$ for the both two sides of equation (17), we have

$$XX^{\mathrm{T}} = (UV^{\mathrm{T}}R)XX^{\mathrm{T}}(UV^{\mathrm{T}}R) \tag{18}$$

Where $XX^{\mathrm{T}} \neq 0$, because the normalized coordinates cannot be all at the centroids. Therefore, for the existence of equation (18), the prerequisite is,

$$UV^{\mathrm{T}}R = (\pm 1)I \tag{19}$$

Where I is 3×3 identity matrix.

From the definition, the determinants of rotation matrix R and orthogonal matrices U and V are $+1$ and ± 1, respectively. So the optimal rotation matrix should be

$$\hat{R}' = \begin{cases} VU^{\mathrm{T}}, \det(VU^{\mathrm{T}}) = 1 \\ -VU^{\mathrm{T}}, \det(VU^{\mathrm{T}}) = -1 \end{cases} \tag{20}$$

Where det denotes the determinant of a matrix.

For compactness, equation (20) can be written as a uniform expression

$$\hat{R} = V\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \det(VU^{T}) \end{pmatrix}U^{\mathrm{T}} \tag{21}$$

Obviously, the optimal rotation matrix is independent to the optimal scale.

In the estimation of $\hat{\lambda}$ and $\hat{R}$, the rotation and scale parameters are irrelative to each other.

Furthermore, we can discuss in details that:

1) If there is only zoom between the two sets of point, matrix S is symmetric, $S = XY^{\mathrm{T}} = \lambda XX^{\mathrm{T}}$ while the SVD is $S = U\Sigma V^{\mathrm{T}} = V\Sigma V^{\mathrm{T}}$. Therefore, the rotation matrix is identity matrix with $\hat{R} = VV^{\mathrm{T}} = I$.

2) All the rotation information is completely included in matrix S, no matter whether the rotation matrix R or quaternion is adopted as the parameterization of rotations. In fact, quaternion-based algorithm still needs to construct matrix S first, then a 4×4 matrix from linear combination the elements of matrix S, and last, do eigen-value decomposition to the 4×4 matrix[7].

3) Matrix S is subject to the random errors in coordinates of the original set of points. However, from the 2-norm of differential of equation (14), we have $\| \delta S \| = \| \delta \Sigma \|$, which indicates SVD would not magnify the impact of noises and can ensure the precision of the estimated rotation matrix[8].

4) It would be convenient to assume that the points and their positional components are independent when the position accuracy of the two sets of points are taken into consideration simultaneously. For example, the weighted coordinates of the two centroids can be refined accounting for the individual positional component precision of the respective set of points in equation (4), whereas 3-dimension position precision of each point, instead of positional components, is weighted to compute the scale in equation (10). Similarly, matrix S can be modified as $S = XMY^{\mathrm{T}}$, where the position errors matrix $M = diag(1/m_1^2 \quad 1/m_2^2 \quad \cdots, 1/m_N^2)$ may be determined by the set of point which is relatively poor in precision, where m_i denotes the position mean square error.

4 Contribution of multi-source data

Currently, besides the pairs of 3-dimension points, the varieties of corresponding information increase gradually with the rapid development of multi-sensors and measurement techniques. For example, relative positioning of GNSS can provide high precision baseline vectors, and a 6-freedom sensor can obtain directly attitude matrix. Therefore, multi-source data fusion is a privilege tool to take full advantage of different types of data regarding their respective contributions.

For example, baseline vectors obtained from relative positioning, linear-character extraction, etc, are sensitive to both rotation and zoom; attitude matrices acquired from 6-freedom sensor-platforms and vision robots, are sensitive to rotation. Whether the above-mentioned analytical approach can be applied to all types of corresponding information analysis, is determined by whether it is sensitive to translation, zoom and rotation, furthermore, the contribution to transformation parameters and calculation methods can be determined.

To utilize these evident contributions to the correlations and solutions of transformation parameters, we should expand the corresponding calculations.

Substituting sub-matrices dX and dX' constituted from pairs of baseline vectors, Ω_1 and Ω_2 attitude matrices and into equation (12), we have extended matrices

$$\begin{aligned} \bar{X} &= (X \quad dX \quad \Omega_1) \\ \bar{Y} &= (Y \quad dX' \quad \Omega) \end{aligned} \tag{22}$$

Then we reconstruct matrix S by $\bar{X}$ and $\bar{Y}$. With the SVD of S, the rotation matrix $\hat{R}$ can be solved.

Similarly, we substitute the length constraints of baseline vectors into equation (10), then

$$\hat{\lambda} = \sqrt{\frac{\sum_{i=1}^{N} |X'_{2i}|^2 + \sum_{j=1}^{M} |dX_j|^2}{\sum_{i=1}^{N} |X'_{1i}|^2 + \sum_{j=1}^{M} |dX'_j|^2}} \tag{23}$$

Finally, we calculate the translation parameters from equation (9).

Note that the weight factors can be attached directly by simultaneously considering the pairs of corresponding information in this approach, whereas it is more complicated using total least square regression[9].

5 Summary

1) In seven-parameter transformation model, the translation parameters have linear relations with the rotation and scale. The estimation of the translation parameters depends strongly on the optimal rotation and scale parameters, whereas the rotation estimates and the scale are irrelative.

2) The normalization with respect to centroid is the key to separate the rotation and zoom from translation. Therefore, we can fit four-parameter transformation model (Euler angles and scale) from the regulated coordinates, and the fit precision is higher than that of seven-parameter transformation model due to the decoupling from the translation on a great extent. Another scheme is commonly used in practice, that is, reduce a considerably large part of the translation quantity with a prior value before fitting seven-parameter transformation model. We can see that it suggests the similar decoupling effect as the centroid-regulation so that it also performs better than the direct seven-parameter fit, the numerical validation of which can be seen in reference [3] for more details.

3) Of all the corresponding pairs of two coordinate systems, 3-dimension points are sensitive to all the transformation motions; baseline vectors are independent from the translation; attitude matrices are merely sensitive to the rotation. We should optimize the solutions of the transformation parameters from multi-source data according to their measurement accuracy and contribution characteristics.

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